Irreducible Graphs

Masanori Koyama
Department of Mathematics, Harvey Mudd College
301 Platt Blvd, Claremont, CA 91711-5990
mkoyama@odin.ac.hmc.edu

David L. Neel
Department of Mathematics, Seattle University
901 12th Avenue, P.O. Box 222000, Seattle, WA 98122-1090
neeld@seattleu.edu

Michael E. Orrison
Department of Mathematics, Harvey Mudd College
301 Platt Blvd, Claremont, CA 91711-5990
orrison@hmc.edu

December 30, 2005

Abstract

A connected graph on three or more vertices is said to be irreducible if it has no leaves, and if each vertex has a unique neighbor set. A connected graph on one or two vertices is also said to be irreducible, and a disconnected graph is irreducible if each of its connected components is irreducible. In this paper, we study the class of irreducible graphs. In particular, we consider an algorithm that, for each connected graph Γ, yields an irreducible subgraph $I(\Gamma)$ of Γ. We show that this subgraph is unique up to isomorphism. We also show that almost all graphs are irreducible. We then conclude by highlighting some structural similarities between $I(\Gamma)$ and Γ.

1 Introduction

This paper is concerned with the class of graphs known as *irreducible graphs*. A connected graph on three or more vertices is *irreducible* if it has no leaves, and if each vertex has a unique neighbor set. A connected graph on one or two vertices is also said to be irreducible, and a disconnected graph is irreducible if each of its connected components is irreducible.

Irreducible graphs played a pivotal role for Neel and Orrison in [3] in which they studied the *linear complexity* of graphs. If Γ is a graph, then its linear complexity $L(\Gamma)$ is a measure of the minimum number of arithmetic operations needed multiply the adjacency matrix of Γ with an arbitrary vector of the appropriate size. It was shown in [3] that if Γ is a connected graph, then it has a connected, induced, and irreducible subgraph $I(\Gamma)$ such that

$$L(\Gamma) = L(I(\Gamma)) + (|V(\Gamma)| - |V(I(\Gamma))|).$$

This fact was then used to find or give bounds for the linear complexity of several well-known classes of graphs.

The construction in [3] of $I(\Gamma)$ is based on an algorithm, which we call the *Reduction Algorithm*, that requires a fixed ordering of the vertices of Γ. In this paper, we show that the isomorphism class of $I(\Gamma)$ is independent of the chosen ordering of the vertices of Γ (Theorem 4). We also show that almost all graphs are irreducible (Theorem 5). We then conclude by highlighting some structural similarities between $I(\Gamma)$ and Γ (Propositions 9, 10, 11).

Throughout the paper, we assume familiarity with the basics of graph theory. See for example [5]. Furthermore, all graphs considered are finite, simple, and undirected. Lastly, we will denote the vertex set of a graph Γ by $V(\Gamma)$, and denote the neighbor set of a vertex v in Γ by $N_\Gamma(v)$.

2 The definition and preliminaries

In [3], a graph Γ was defined to be *irreducible* if it was a connected graph that remained unchanged after undergoing the following, which we call the *Reduction Algorithm*:

Let Γ be a connected graph on at least three vertices with vertex set $V(\Gamma) \subseteq \{v_1, \ldots, v_n\}$. Let $R(\Gamma)$ denote the induced subgraph of Γ obtained by removing the vertex $v_j \in V(\Gamma)$ with the smallest index j such that

1. v_j is a leaf, or
2. there exists some $v_i \in V(\Gamma)$ such that $i < j$ and $N_\Gamma(v_j) = N_\Gamma(v_i)$.

If no such vertex exists, then we define $R(\Gamma)$ to be Γ. For convenience, we also define $R(\Gamma) = \Gamma$ if Γ consists of only one edge or one vertex.

When this single-element reduction is repeated we construct, in the process, a sequence of induced subgraphs: $\Gamma, R(\Gamma), R^2(\Gamma), \ldots$. Let k be the first natural number such that $R^{k+1}(\Gamma) = R^k(\Gamma)$. We define $I(\Gamma)$ to be $R^k(\Gamma)$.

It is clear that $I(\Gamma)$ is connected, contain no leaves, and the neighbor sets of distinct vertices will be distinct. It is therefore an irreducible induced subgraph of Γ. In the context of [3], Neel and Orrison focused primarily on connected graphs. The definition of irreducible can be naturally extended to disconnected graphs: we call a disconnected graph irreducible if each of its connected components is irreducible.

Note that in an irreducible graph all vertices v with $deg(v) \geq 1$ will have distinct neighbor sets, and the only leaves will be in connected components consisting of two vertices connected by an edge.

We may easily adapt the Reduction Algorithm to disconnected graphs by applying the algorithm to each connected component individually. Because irreducibility questions which involve disconnected graphs rest entirely on associated irreducibility questions involving the connected components, we will restrict our focus to connected graphs for the remainder of this paper.

We define an equivalence relation on the vertices of a connected graph in the following way: Vertices v and w are equivalent, $v \sim w$, if $N_\Gamma(v) = N_\Gamma(w)$. The equivalence classes will be sets of vertices with the same neighbor sets in Γ. We denote the equivalence class containing a vertex v by $[v]_\Gamma$, or simply $[v]$ when the graph in question is unambiguous. For any nontrivial equivalence class $[v]$, we call the deletion of a nonempty set $S \subset [v]$ from $V(\Gamma)$ a *reduction* of the equivalence class $[v]$. We will say that an equivalence class $[v]$ is *reducible* if $|[v]| > 1$. We say a vertex v is reducible if it is a leaf or $[v]$ is reducible.

3 Uniqueness and Prevalence

The Reduction Algorithm depends on the ordering of the vertices of the graph. A natural question is whether this ordering affects the form of the irreducible graph $I(\Gamma)$. We will show that, in fact, $I(G)$ is unique up to isomorphism. We will begin with a series of useful lemmas.

Lemma 1. *Let Γ be a graph, $R(\Gamma)$ a single-element reduction of the graph, and let $u, v \in V(R(\Gamma))$. If $u \sim v$ in Γ, then $u \sim v$ in $R(\Gamma)$.*

Proof. Since $u \sim v$ in Γ, $N_\Gamma(u) = N_\Gamma(v)$, and $R(\Gamma)$ is formed by deleting a single vertex. Clearly, $N_{R(\Gamma)}(u) = N_{R(\Gamma)}(v)$, so $u \sim v$ in $R(\Gamma)$. $\square$

This yields a second lemma concerning equivalence classes as a corollary:

Lemma 2. *Let Γ be a graph, $R(\Gamma)$ a single-element reduction of Γ, and $u \in V(R(\Gamma))$. If $[u]$ was not the equivalence class that was reduced, then $[u]_\Gamma \subseteq [u]_{R(\Gamma)}$.*

The proof is immediate. A final useful lemma follows:

Lemma 3. *Let Γ be a graph. Let $u \in V(R(\Gamma))$ be a reducible vertex in Γ. If $[u]$ was not the equivalence class that was reduced, then u is a reducible vertex in $R(\Gamma)$.*

Proof. If u is a leaf in Γ, it will still be a leaf in $R(\Gamma)$ since its sole neighbor cannot have been a reducible vertex. If $[u]_\Gamma$ was reducible, then so is $[u]_{R(\Gamma)}$ by Lemma 2. $\square$

With these lemmas in hand we may prove the desired result regarding the uniqueness of $I(\Gamma)$.

Theorem 4. *For any graph Γ, the irreducible graph $I(\Gamma)$ is unique up to isomorphism.*

Proof. Suppose, for the sake of contradiction, that there exists a minimal counterexample Γ. Then Γ is a minimal reducible graph containing a pair of reducible vertices u, v such that $I(\Gamma - u) \ncong I(\Gamma - v)$. Note that we may speak without ambiguity of the irreducible subgraphs of $\Gamma - v$ and $\Gamma - u$ since Γ is minimal. Clearly, $[u] \neq [v]$.

By Lemma 3, v is reducible in $\Gamma - u$, as is u in $\Gamma - v$. Thus, $\Gamma - u$ may be reduced by deleting v, as can $\Gamma - v$ by deleting u. In each case this reduction results in the graph $\Gamma - \{u, v\}$. It follows that $I(\Gamma - u) = I(\Gamma - \{u, v\}) = I(\Gamma - v)$, which is a contradiction. Thus, no minimal counterexample exists, which completes the proof. $\square$

Armed with this theorem, we may now speak of the irreducible subgraph $I(\Gamma)$ without ambiguity, up to isomorphism.

Another interesting point to consider is whether irreducible graphs are common, rare, or something in between. In other words, when plucking, at random, some graph from the vast universe of possible graphs, can we say anything interesting about the likelihood of that graph being irreducible? We now show that most graphs, indeed, nearly all graphs, are irreducible.

Let Γ be a graph, and let u and v be distinct vertices of Γ. If $N_\Gamma(u) = N_\Gamma(v)$, then there exists a nontrivial automorphism of Γ, namely the automorphism that simply transposes u and v. A graph that has no nontrivial autormorphisms is said to be *asymmetric*. It therefore follows that every leafless asymmetric graph must be irreducible. Using this fact, we may prove the following theorem:

Theorem 5. *Almost all graphs are irreducible, i.e., the proportion of graphs on n vertices that are irreducible goes to 1 as $n \to \infty$.*

Proof. It is well known that almost all graphs are asymmetric (see, for example, Corollary 2.3.3 in [2]). On the other hand, it was shown in [4] that if k is any fixed positive integer, then almost all graphs are k-connected. It follows, in particular, that almost all graphs are asymmetric and 2-connected. Since all such graphs are leafless and asymmetric, and hence irreducible, this completes the proof. $\square$

4 Structural characteristics of the irreducible subgraph

We now consider what, if anything, is structurally preserved during the application of the Reduction Algorithm.

Proposition 6. *Let Γ be a graph with a connected irreducible induced subraph H. Then $I(\Gamma)$ contains an isomorphic copy of H as an induced subgraph.*

Proof. By Theorem 4 we know that $I(\Gamma)$ is unique up to isomorphism. In particular, $I(\Gamma)$ does not depend upon the ordering of the vertices used by the Reduction Algorithm.

Thus, we may order the vertices of Γ such that the vertices $\{v_1, \ldots, v_k\} = V(H)$ are the first k vertices. We now show that $V(H) \subseteq V(I(\Gamma))$ by arguing that no v_i can be removed by the Reduction Algorithm with this vertex ordering.

We proceed by contradiction. Suppose v_j is the first vertex of H removed by the Reduction Algorithm. Since H is irreducible, we know that such a v_j is not a leaf. Thus it must have been removed as part of a reducible equivalence class; that is, $[v_j]$ is reducible at this stage of the algorithm. This implies there must be some vertex $w \in [v_j]$ listed before v_j in the ordering of the vertices. Thus $w \in V(H)$, and we know that $N_H(w) \neq$

$N_H(v_j)$ since H is irreducible. Since no vertices from H have yet been removed, $N(w) \neq N(v_j)$ at this stage of the algorithm, which forces $w \notin [v_j]$, a contradiction.

Since no vertex of $V(H)$ can be removed by the Reduction Algorithm for this ordering of vertices, H must be a connected induced subgraph of $I(\Gamma)$. Thus, for any ordering of the vertices, $I(\Gamma)$ must contain a connected induced subgraph isomorphic to H, by Theorem 4. $\square$

The above result may be extended, but first we need a definition. Recall that the distance $d(u,v)$ between two vertices u, v in a graph is the length of the shortest path between them.

Definition 7. *Given a graph Γ, and induced subgraphs H_1, H_2 of Γ, we say H_1 and H_2 are* separable *if and only if* $\min\{d(u,v) : u \in H_1, v \in H_2\} \geq 2$.

In the following theorem, we insist that the irreducible subgraphs have order 3, since otherwise we are commenting on single vertices or edges within Γ and $I(\Gamma)$. In this case, we are interested in the possible persistence of larger irreducible subgraphs in $I(\Gamma)$, and their interactions or lack thereof.

Theorem 8. *Let $H_1, \ldots, H_m$ be connected irreducible induced subgraphs of a graph Γ that are pairwise separable and have order at least 3. Then $I(\Gamma)$ contains pairwise separable subgraphs isomorphic to $H_1, \ldots, H_m$.*

Proof. Let $V(H_1) \cup \cdots \cup V(H_m) = \{v_1, \ldots, v_k\}$. As before, we order $V(\Gamma)$ such that $\{v_1, \ldots, v_k\}$ are the first k vertices. And also as before, we will show that no v_i can be removed by the Reduction Algorithm with this ordering, and then appeal to Theorem 4.

We proceed by contradiction, and let v_j be the first vertex in $\{v_1, \ldots, v_k\}$ removed by the Reduction Algorithm. Since all H_i are irreducible, we know v_j cannot be a leaf, and thus must have been removed as a member of the reducible equivalence class, $[v_j]$. But, as above, this means there exists some $w \in \{v_1, \ldots, v_{j-1}\} \cap [v_j]$. We saw in the proof of the previous proposition that w cannot be in the same H_i as v_j. So $w \sim v_j$, and they are contained in separable subgraphs H and H' respectively, $H, H' \in \{H_1, \ldots, H_m\}$. But $N_\Gamma(w) = N_\Gamma(v_j)$ forces $H \cap H' \neq \emptyset$ since H and H' are connected graphs of order at least 3. This contradicts the fact that H and H' are separable, and thus no $v_i \in \{v_1, \ldots, v_k\}$ can be removed. This proves that the subgraphs $H_1, \ldots, H_m$ are present in $I(\Gamma)$.

We recall that $I(\Gamma)$ is just an induced subgraph of Γ, and thus $d_\Gamma(u,v) \leq d_{R(\Gamma)}(u,v)$ for all $u, v \in V(R(\Gamma))$. So the irreducible subgraphs $H_1, \ldots, H_m$, which are pairwise separable in Γ, remain pairwise separable in $I(\Gamma)$. $\square$

5 Three observations and some open questions

We will close with three observations about how the irreducible graph $I(\Gamma)$ is closely related to certain graph characteristics of Γ, followed by a brief discussion of open questions.

First we consider the chromatic number $\chi(\Gamma)$:

Proposition 9. *Let Γ be a connected graph. Then $\chi(\Gamma) = \chi(I(\Gamma))$.*

Proof. If Γ is irreducible, the result is immediate. We will show $\chi(\Gamma) = \chi(R(\Gamma))$ for any reducible connected graph Γ. This will prove the theorem.

Since $R(\Gamma)$ is an induced subgraph of Γ, it is immediate that $\chi(R(\Gamma)) \leq \chi(\Gamma)$.

Now suppose $R(\Gamma) = \Gamma - v$, and c is a proper coloring of $R(\Gamma)$. Either v was a leaf, or v was a member of a reducible equivalence class. In either case, we can build a coloring of Γ from a c with no additional colors.

First, suppose v was a leaf in Γ and $N_\Gamma(v) = w$. Then to color Γ, color all the vertices as they were colored by c in $R(\Gamma)$, and color v with any color other than the color c assigned to w. This is clearly a proper coloring.

Second, suppose $[v]_\Gamma$ was a reducible equivalence class. Then to color Γ, color v with the color assigned to some other vertex $w \in [v]_\Gamma - \{v\}$ by coloring c of $R(\Gamma)$. This is a proper coloring of Γ since the $[v]_\Gamma$ is an independent set of vertices, thus $w \notin N_\Gamma(v)$, and no vertices in $N_\Gamma(w) = N_\Gamma(v)$ use the color assigned to w by c.

Thus, $\chi(\Gamma) \leq \chi(R(\Gamma))$ for any Γ, and therefore $\chi(\Gamma) = \chi(R(\Gamma))$. It follows that $\chi(\Gamma) = \chi(I(\Gamma))$. $\square$

Next we consider the girth of these graphs. Recall that $girth(\Gamma)$ is the length of the shortest cycle in a graph.

Proposition 10. *Let Γ be a graph with girth $k \neq 4$. Then $girth(I(\Gamma)) = k$.*

Proof. Since $I(\Gamma)$ is an induced subgraph of Γ, it is immediate that $k \leq girth(I(\Gamma))$.

Now, an isolated cycle of size $k \neq 4$ is irreducible since neighbor sets are unique. (The uniqueness of neighbor sets fails for cycles of length 4.)

Let C be a cycle of minimal length k in Γ. Clearly C must be an induced subgraph of Γ. This, combined with the irreducibility of C implies that a graph isomorphic to C is present in $I(\Gamma)$ by Theorem 6. Thus $girth(I(\Gamma)) \leq k$, which completes the proof. □

Proposition 11. *Let Γ be a graph, $girth(\Gamma) = 4$. Then either $girth(I(\Gamma)) = 4$, $girth(I(\Gamma)) = \infty$, or $girth(I(\Gamma))$ is equal to the length of a minimal cycle among all other cycles of Γ.*

Proof. There are two possibilities for each cycle of length 4 in Γ. The first possibility is that the cycle will, because of adjacencies to other vertices, be present in $I(\Gamma)$. The second possibility for each 4-cycle is that it eventually becomes isolated enough that the Reduction Algorithm removes one or more vertices from it.

Again, $I(\Gamma)$ is an induced subgraph of Γ, so $girth(\Gamma) \leq girth(I(\Gamma))$. Thus, if one or more 4-cycles is contained in $I(\Gamma)$ then the girth remains 4.

Suppose all 4-cycles are broken or removed during reduction. If there are any other cycles in Γ, they must necessarily have length greater than 4. Let C be the minimal cycle of length $k > 4$ in Γ. Cycles of length greater than 4 are irreducible. C is therefore an irreducible subgraph of Γ, and by Theorem 6 there must be an induced subgraph of $I(\Gamma)$ isomorphic to C. Thus the $girth(I(\Gamma)) = k$.

If no cycles remain after the Reduction Algorithm (that is, if all cycles in Γ had length 4 and were then reduced), then, by definition, $girth(I(\Gamma)) = \infty$. □

Other graph characteristics seem less hereditary, including both vertex- and edge-connectivity. Still, one suspects there are still things to be said about the relationship between Γ and $I(\Gamma)$ and how it interacts with these and other graph characteristics.

It may also be fruitful to investigate whether the Reduction Algorithm and some similar conclusions might be adapted to directed graphs.

Finally, we are interested in investigating how close an irreducible graph is to being reducible, or vice versa. Construct the graph of all graphs on n vertices, $\mathcal{G}$, in the following way: Let $V(\mathcal{G})$ be the set of all graphs on n vertices. We say two graphs on n vertices are adjacent in $\mathcal{G}$ if their edge sets differ by only one edge. Thus, for any Γ on n vertices, $N_{\mathcal{G}}(\Gamma)$ will be all graphs that can be formed from Γ by the addition or deletion of a single edge. By Theorem 5 we know that as n grows, nearly all vertices in $\mathcal{G}$ will be irreducible graphs, so perhaps it would be most interesting to investigate the distance from an arbitrarily selected irreducible graph to the nearest reducible graph. Is it the case that, despite how rare reducible graphs are, each irreducible is yet quite close to a reducible graph in $\mathcal{G}$?

6 Acknowledgments

We gratefully acknowledge a Harvey Mudd College Beckman Research Award which supports research by undergraduates with faculty. Special thanks also to Ross Richardson for directing our attention to the results found in [4].

References

[1] P. Bürgisser, M. Clausen, and M.A. Shokrollahi, *Algebraic complexity theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Springer-Verlag, Berlin, 1997, With the collaboration of Thomas Lickteig.

[2] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

[3] D. Neel and M. Orrison, *The linear complexity of a graph* (preprint).

[4] T. R. S. Walsh and E. M. Wright, *The k-connectedness of unlabelled graphs*, J. London Math. Soc. (2) **18** (1978), no. 3, 397–402.

[5] D. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996.