# Soln key for 2004 Fall Math 221 Prof Passman's exam 2

November 13, 2007

## 1 a)

This is just implicit differentiation. The given expression yields

$$2x - \left(y + \frac{dy}{dx} \cdot x\right) + 2y\frac{dy}{dx} = 0$$

You plus in $(5, 1)$ to the equation above. Then

$$\begin{aligned} 10 - \left(1 + \frac{dy}{dx} \cdot 5\right) + 2\frac{dy}{dx} &= 0 \\ 9 - 3\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= 3. \end{aligned}$$

## 1 b)

Again, this is just an implicit differentiation. We get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\frac{1}{x} + y\left(-\frac{1}{x^2}\right)$$

Plug in the given expression of $\frac{dy}{dx}$ in to the equation above. In particular,

$$\frac{d^2y}{dx^2} = \left(\frac{y}{x} + 3\right)\frac{1}{x} + y\left(-\frac{1}{x^2}\right).$$

Simply as much as you want to your taste.

## 2 a)

The key is to recognize similar triangles. In particular, note that in the given picture, the bit triangle and the small triangle with base $|TM|$ are similar. Suppose that the distance from the dude to the lamp is given by $y$. Then notice

$$\frac{opposite}{base} = \frac{6}{|TM|} = \frac{16}{|TM| + y}.$$

Therefore we get

$$\begin{aligned} 6(|TM|+y) &= 16|TM| \\ 6y &= 10|TM| \\ \frac{dy}{dt} &= \frac{5}{3}\frac{d}{dt}|TM| \\ 5ft/sec &= \frac{5}{3}\frac{d}{dt}|TM| \\ 3ft/sec &= \frac{d}{dt}|TM|. \end{aligned}$$

## 2 b)

Notice $\tan\theta = \frac{6}{|TM|}$. Note that when $y = 10$, $|TM| = 6$. Thus $\theta = \pi/4$. Therefore

$$\begin{aligned} \frac{d\theta}{dt}\sec^2(\theta) &= -\frac{16}{|TM|^2}\frac{d}{dt}|TM| \\ \frac{d\theta}{dt}\sec(\pi/4) &= -\frac{16}{6^2} * 3 \end{aligned}$$

You can figure out $\frac{d\theta}{dt}$ easily from the relation above.

## 3

Note that the rectangle ABCD is controlled by only one parameter—-the angle $\theta$ of the triangle $OCD$ (Not Obsessive complusive Disorder!) where $O$ is the origin. Note that the hypotenuse of this triangle has the length of radius of the circle, which is $\sqrt{5}$. In particular, the perimeter of the ABCD when the angle between $OC$ and $OD$ is $\theta$ is :

$$\begin{aligned} P_{ABCD} &= 2 * \underbrace{|CD|}_{\text{height}} + 2 * \underbrace{(|OD| + |OA|)}_{\text{width}} \\ &= 2\underbrace{\sqrt{5}\sin(\theta)}_{\text{height}} + 2 * \underbrace{2\sqrt{5}\cos(\theta)}_{\text{width}} \end{aligned}$$

(If you are like "Whata hell is maso talking about!" then and draw each of what I am saying into picture.) The range of the $\theta$ we consider is, indeed $0 \leq \theta \leq \pi/2$. Now

$$\frac{d}{dt}P_{ABCD} = 2\sqrt{5}(\cos\theta - 2\sin\theta)$$

This evaluates to zero when $\cos\theta - 2\sin\theta = 0$, that is, when

$$|OD| = 2 * |CD|,$$

(Again, draw the triangle $OCD$ if you are lost here.) Then by pythagorean theorem, for such size of $OD$,

$$\begin{aligned} 5 &= (2 * |CD|)^2 + |CD|^2 \\ &= 4|CD|^2 + |CD|^2 \\ &= 5|CD|^2 \end{aligned}$$

and $|CD| = 1$, and $2 = |OD|$ Thus the perimeter for such length of $OD$ is $2 + 4 * 2 = 10$. This is bound to be max because the boundary value of $\theta$ gives $2\sqrt{5}$ (when $\theta = 0$ and $\pi/2$), which is smaller than our 8.

## 4 a)

This should be cake for you. The derivative of $y$ is $4x^2(x-3)$. Because $x^2$ is always positive, the sign of this is controlled solely by $(x-3)$. The function is increasing for $x \geq 3$, decreasing for $x < 3$. Local minima at 3.

## 4 b)

The second derivative is $12x(x-2)$. The sign analysis should lead you to conclude that the graph is concave down in $0 < x < 2$ and concave up everywhere. The inflection points are indeed $x = 0$ and $x = 2$.

## 4 c)

I forgot how to import the pictures into tex file. So I will just remind you that this is only about pasting of "parabolas" that I did in discussion. First, locate $y$ intersept(just plug in 0 in to y), and zeros of the function. Put that into graph. Then do that pasting "parabolas" using the information given in $b$) and $c$).

## 5 a)

First, recall the mean value theorem. Then you should see that you are looking for a point on the graph such that the derivative at that point is

$$\frac{f(4)-f(1)}{4-1}.$$

Thus you are simply solving

$$3x^2 - 12x = \frac{f(4)-f(1)}{4-1} = 9.$$

The solution to this quadratic is the answer.

## 5 b)

Put $u = x^2 + 2x + 3$. Then $du = 2(x+1)dx$. So the integral can be written as

$$\begin{aligned}\int (x+1)\sqrt{u}\frac{du}{2(x+1)} &= \int \sqrt{u}\frac{du}{2} \\ &= \frac{1}{2}\frac{2}{3}u^{3/2} + C \\ &= \frac{1}{3}u^{3/2} + C.\end{aligned}$$

## 6 a)

Caution: you don't have to be able to do this for the upcoming exam.

This is a separable differential equation. In particular,

$$\begin{aligned} \frac{dy}{dx} &= 2xy^2 \\ \frac{dy}{y^2} &= 2xdx \\ \int \frac{dy}{y^2} &= \int 2xdx \\ -\frac{1}{y} &= x^2 + C \end{aligned}$$

You figure out the $C$ with the initial condition $(0, 1)$. By plugging $x = 0, y = 1$, you see that $C = 1$. Thus

$$-\frac{1}{y} = x^2 + 1,$$

or

$$y = -\frac{1}{x^2 + 1}.$$

## 6 b)

You separate them into pieces. The first piece,

$$\sum_{i=1}^{n} 2^{i+1} - 2^i$$

is the telescopic sum that I gave you in the discussion. You should see from the technique I gave you that this evaluates to $2^{n+1} - 2$. On the other hand,

$$\sum_{i=1}^{n} 5a_i = 5 * 4 = 20 \qquad \text{and} \qquad \sum_{i=1}^{n} 2b_i = 2 * 9 = 18.$$

Thus the total sum is

$$2^{n+1} - 2 + 20 - 18 = 2^{n+1}.$$