Soln key for 2004 Fall Math 221 Prof Passman's exam

October 8, 2007

Tips about exam:

- Make sure the the writing is legible.
- Make sure you explain each step of what you are doing.
- If you are lost, then write what you wanted to do clearly in complete sentences.
- Don't do scratch work in the soln space.
- Read over your answer at least once before proceeding to the next question.

1 a)

Recall that "a point" and a "slope" determines a line. In particular, if a line go through a point (a, b) and has a slope m, then the line is expressed by

$$m(x-a) = (y-b).$$

You can visualize this if you think of what $\frac{y-b}{x-a}$ mean. (Do think!)

Because the given line has a slope 2, the line that we are cooking must have the slope $-\frac{1}{2}$. Moreover, this point is going through $(6, 1)$. Putting these info into the form above, we get

$$\begin{aligned} -\frac{1}{2}(x-6) &= (y-1) \\ y &= -\frac{1}{2}x + 4. \end{aligned}$$

We want to know where this line and $2x - 1$ crosses. So all we need to do is

$$\begin{aligned} 2x - 1 &= -\frac{1}{2}x + 4 \\ \frac{5}{2}x &= 5 \\ x &= 2. \end{aligned}$$

Therefore they cross at $(2, 2*2-1) = (2, 3)$. Now what is the distance from p to the original line? We saw that, if we draw from p a perpendicular line to the original line, the line will cross at $(2, 3)$. So the distance is given by the length of the segment from p to $(2, 3)$. This, we can compute. The distance d will be given by

$$\sqrt{|6-2|^2 + |1-3|^2} = \sqrt{16+4} = 2\sqrt{5}.$$

1 b)

The canonical form of the circle with radius r and the center (a, b) is given by

$$(x-a)^2 + (y-b)^2 = r^2$$

So let's get the equation into this form by completing the square.

$$\begin{aligned}
x^2 + y^2 - 4x + 2y - 20 &= 0 \\
x^2 - 4x + y^2 + 2y &= 20 \\
(x-2)^2 - 4 + (y+1)^2 - 1 &= 20 \\
(x-2)^2 + (y+1)^2 &= 25 \\
(x-2)^2 + (y+1)^2 &= 5^2
\end{aligned}$$

Thus the radius is 5 and the center is $(2, -1)$.

How do I get the diametrically opposite point? Try drawing the picture. You shall see when you draw that the diametrically opposite point of (x, y) in this circle is given by

$$(2 - (x-2), -1 - (y - (-1)))$$

Thus the diametrically opposite point of $(-1, 3)$ is given by $(5, -5)$.

2 a)

You do something that is very similar to what I did in the class. Suppose that ϵ is given. Then choose δ so that $\delta < \epsilon/12$ and $\delta < 1$. You can check by drawing on the real line that, If I let $|x-3| < \delta$ for such δ, then $2 < x < 4$(because the distance from x to 3 will be smaller than δ, which is smaller then 1!) so that $|x+8| < 12$. Thus, for such choice of δ,

$$\begin{aligned}
|(x^2 + 5x + 1) - 25| &= |x^2 + 5x - 24| \\
&= |x-3||x+8| \\
&< (\epsilon/12)(12) \\
&= \epsilon
\end{aligned}$$

as desired. $\square$

2 b)

Clearly, the bottom $(x^2 - 4)$ is a pain in the ass. In particular,

$$x^2 - 4 = (x-2)(x+2)$$

and $(x-2)$ part is giving us a hard time, because $x - 2$at $x = 2$ is 0. Well, we want to get rid of it certainly. So first thing you should check is if we can factor out the renegade $(x-2)$ from the top. And it turns out that we can!!

$$\begin{aligned}
\lim_{x\to 2}\frac{x^3-x^2+4x-12}{x^2-4} &= \lim_{x\to 2}\frac{(x^2+x-6)(x-2)}{(x-2)(x+2)}\\
&= \lim_{x\to 2}\frac{x^2+x-6}{x+2}\\
&= \frac{4+2-6}{4}\\
&= 0
\end{aligned}$$

Note that, in the second line I was able to plug in the number into the expression because polynomial is continuous... that is,

$$\lim_{x\to a} x^2+x-6 = a^2+a-6.$$

3 a)

This might look like it is a pain, but it is not. When you see a complicated limit, then first thing to do is to factor out.

$$\begin{aligned}
\lim_{x\to 0}\frac{xsin(x)}{cos(x)-cos^2(x)} &= \lim_{x\to 0}\frac{xsin(x)}{cos(x)(1-cos(x))}\\
&= \lim_{x\to 0}\frac{xsin(x)}{cos(x)(1-cos(x))}\frac{(1+cos(x))}{(1+cos(x))}\\
&= \lim_{x\to 0}\frac{xsin(x)(1+cos(x))}{cos(x))(1-cos^2(x))}\\
&= \lim_{x\to 0}\frac{xsin(x)(1+cos(x))}{cos(x)(1-cos^2(x))}\\
&= \lim_{x\to 0}\frac{xsin(x)(1+cos(x))}{cos(x)(sin^2(x))}\\
&= \lim_{x\to 0}\frac{x(1+cos(x))}{cos(x)(sin(x))}\\
&= \lim_{x\to 0}\frac{(1+cos(x))}{cos(x)}\lim_{x\to 0}\frac{x}{(sin(x))}\\
&= 2
\end{aligned}$$

3 b)

To be continuous, we want $\lim_{x\to a} f(x) = f(a)$. We know that, regardless of the value of c, the given function is continuous evereywhere except at 2 because each piece of this piecewise function is polynomial. Therefore we want

$$\lim_{x\to 2^-} 5x+c = \lim_{x\to 2^+} x^2+3c$$

In short, we are good if

$$5*2+c = 2^2+3c$$

Thus $c = 3$ suffices.

4 a)

This is just a computation nightmare. Patience is the key!

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{(x^2+1}}{h} &= \lim_{h\to 0} \frac{((x+h)^2+1)-(x^2+1)}{h\sqrt{(x+h)^2+1} + \sqrt{(x^2+1}} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h\sqrt{(x+h)^2+1} + \sqrt{(x^2+1}} \\
&= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \\
&= \frac{2x}{\sqrt{x^2+1} + \sqrt{x^2+1}} \\
&= \frac{x}{\sqrt{x^2+1}}
\end{aligned}$$

In the exam, make sure to check your answer with the chain rule.

4 b)

Well, first get the slope of the tangent line. This is just a derivative of this function at $x = 1$. By quotient rule, derivative is

$$\frac{2x(x+1) - x^2 * 1}{(x+1)^2}.$$

And this evaluated at $x = 1$ is

$$\frac{2(1+1) - 1 * 1}{(1+1)^2} = \frac{3}{4}.$$

Next, I want to know what

$$\frac{x^2}{x+1}$$

evaluates at $x = 1$, because the tangent line must touch the graph of this function at $x = 1$. We can easily check this to be $1/2$. Thus the tangent line is given by

$$(y - \frac{1}{2}) = \frac{3}{4}(x-1).$$

Done! $\Box$

5 a)

Use the trick that I taught you in the class (the one about taking the derivative of a product of multiple functions). With that you can get away from doing the product rule twice, which might make you prone to make a careless mistake.

$$f'(x) = 1(x+1)^{-4}(x+2)^3 + x\Big((-4)(x+1)^{-5}\Big)(x+2)^3 + x(x+1)^{-4}\Big(3(x+2)^2\Big)$$

After that, just simplify as much as you can...

5 b)

This is just a quotient rule. Hi di ho ho di hi.

$$f'(x) = \frac{\sec^2(x)(1+\sec x) - \tan x(\sec x \tan x)}{(1+\sec x)^2}$$

Again, just simplify as far as you can go.

6 a)

Decompose into two functions. If $g(x) = x^7$ and $h(x) = \frac{sin(x)}{1+x^2}$, then $f(x) = g(h(x))$ and

$$\begin{aligned} f'(x) &= g'(h(x))h'(x) \\ &= 7(h(x))^6 * \frac{\cos x(1+x^2) - \sin x(2x)}{(1+x^2)^2} \\ &= 7\Big(\frac{sin(x)}{1+x^2)}\Big)^6 * \frac{\cos x(1+x^2) - 2x \sin x}{(1+x^2)^2} \end{aligned}$$

6 b)

Decompose into three functions. If $g(x) = \cos x$, $h(x) = x^5$, $k(x) = x^3 + \sin 2x$, then $f(x) = g(h(k(x)))$. So

$$\begin{aligned} f'(x) &= g'(h(k(x))) * h'(k(x)) * k'(x) \\ &= -\sin\left[(x^3 + \sin 2x)^5\right] * 5(x^3 + \sin 2x)^4 * (3x^2 + 2\cos 2x) \end{aligned}$$

This solution key was prepared by Masanori Koyama.